The McKay-Thompson series of Mathieu Moonshine modulo two

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Abstract

In this note, we describe the parity of the coefficients of the McKay-Thompson series of Mathieu moonshine. As an application, we prove a conjecture of Cheng, Duncan and Harvey stated in connection with umbral moonshine for the case of Mathieu moonshine.

1 Introduction

In 2010, Eguchi, Ooguri, and Tachikawa [9] discovered a phenomenon connecting the Mathieu group M_{24} and the elliptic genus of a K3 surface. To describe their observation, we let $q = e^{2\pi i\tau}$ and introduce the following functions:

$$\begin{aligned}\theta_1(z;\tau) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i(n+\frac{1}{2})(z+\frac{1}{2})},\\ \mu(z;\tau) &= \frac{ie^{\pi iz}}{\theta_1(z;\tau)} \sum_{n\in\mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi inz}}{1-q^n e^{2\pi iz}},\end{aligned}$$

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and we set $\Sigma(\tau)$ as follows:

$$\begin{aligned}\Sigma(\tau) &= -8(\,\mu(1/2;\tau) + \mu(\tau/2;\tau) + \mu((\tau+1)/2;\tau) \\ &= q^{-\frac{1}{8}}(-2 + 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + \cdots).\end{aligned}$$

The Mathieu moonshine phenomenon is that the first five coefficients appearing in the Fourier expansion divided by 2,

$$\{45, 231, 770, 2277, 5796\},$$

are equal to dimensions of irreducible representations of M_{24} and further coefficients can be written as simple sums of dimensions of the irreducible representations of M_{24}, for example $13915 = 3520 + 10395$. The reason for this mysterious phenomenon is still unknown.

This observation suggested the existence of a virtual graded M_{24}-module $K = \bigoplus_{n=-1}^{\infty} K_n\, q^{n/8}$ such that for $n \geq 0$ the K_n are honest M_{24}-representations. In analogy to the monstrous moonshine case [5], one can consider for an element g in the conjugacy class ℓX of M_{24} the so-called McKay-Thompson series

$$\Sigma_{\ell X}(\tau) = \sum_{n=-1}^{\infty} \mathrm{Tr}(g|K_n)\, q^{n/8}.$$

In [11, 7] (cf. also [1, 10, 2]), candidates for the 26 McKay-Thompson series for the Mathieu moonshine have been proposed. We note that the McKay-Thompson series for the conjugacy classes in the pairs $(7A, 7B)$, $(14A, 14B)$, $(21A, 21B)$, $(15A, 15B)$ and $(23A, 23B)$ are equal to each other. We list in the appendix the 21 different McKay-Thompson series. Using explicit formulas it can be shown that all the McKay-Thompson series have integer coefficients.

Since the traces for all conjugacy classes together determine the M_{24}-representation, the Mathieu moonshine as now proven by Terry Gannon [12] can more precisely be formulated as follows:

Theorem 1.1 (Mathieu moonshine module)**.** *The McKay-Thompson series as in [11, 7] determine a virtual graded M_{24}-module $K = \bigoplus_{n=-1}^{\infty} K_n\, q^{n/8}$. For $n \geq 0$ the K_n are honest (and not only virtual) M_{24}-representations. Furthermore, each irreducible M_{24}-representation λ occurs in K_n as member of a pair $\lambda \oplus \bar{\lambda}$.*

This implies that if a irreducible representation λ is real, i.e. $\lambda = \bar{\lambda}$, then the multiplicity of λ in K_n is even.

To illustrate our main result, we consider the McKay-Thompson series for $7A$:

$$\begin{aligned}\Sigma_{7A}(\tau) =& \frac{1}{8}\left(\Sigma(\tau)\eta(\tau)^3 - 14\,\phi_2^{(7)}(\tau)\right)\Big/\eta(\tau)^3 \\ =& -2q^{-1/8} - q^{7/8} + 4q^{31/8} - 2q^{47/8} + 2q^{55/8} - 3q^{63/8} + 6q^{87/8} - 6q^{103/8} \\ & - 4q^{119/8} + 8q^{143/8} - 6q^{159/8} + 4q^{167/8} - 7q^{175/8} + 12q^{199/8} - 10q^{215/8} \\ & + 2q^{223/8} - 6q^{231/8} + 18q^{255/8} - 10q^{271/8} + 6q^{279/8} - 12q^{287/8} \\ & + 20q^{311/8} - 18q^{327/8} + 6q^{335/8} - 13q^{343/8} + 28q^{367/8} + \cdots .\end{aligned}$$

One observes that the coefficient of $q^{n/8}$ in $\Sigma_{7A}(\tau)$ is odd if $n = 7m^2$, where m is odd. In general we show:

Theorem 1.2. *For a conjugacy class ℓX of M_{24}, the coefficient of $q^{n/8}$ in $\Sigma_{\ell X}(\tau)$ is odd if and only if $\ell X \in \{7A, 7B, 14A, 14B, 15A, 15B, 23A, 23B\}$ and $n = \ell m^2$, where m is odd or $\ell X \in \{21A, 21B\}$ and $n = \ell m^2$, where m is odd and m is not divisible by* 3.

In this paper, we study the congruences of the Fourier coefficients of the McKay-Thompson series modulo two. For congruences for other primes, for $\Sigma(\tau)$ we refer to the references [14, 16] and for the McKay-Thompson series, the answer will be given in the near future by one of the authors [15]. The reason for considering the case of modulo two is that it explains the appearance of certain irreducible representations of M_{24}. The following conjecture was made in [3], which we state for the case of the Mathieu moonshine only.

Conjecture 1.1 ([3], Conj. 5.11)**.** *Let $n = \ell m^2 \equiv 7 \pmod 8$. Then the M_{24}-representation K_n determined by the coefficients of $q^{n/8}$ of the McKay-Thompson series contains the following conjugate pairs of irreducible representations:*

- *For $\ell = 7$, one of the pairs (χ_3, χ_4), (χ_{12}, χ_{13}) or (χ_{15}, χ_{16});*
- *for $\ell = 15$, the pair (χ_5, χ_6);*
- *for $\ell = 23$, the pair (χ_{10}, χ_{11}).*

Here χ_i denotes the i-th irreducible representation as listed in the ATLAS [4].

Using Theorem 1.2, one has now the following result which includes the above conjecture.

Theorem 1.3. *Let $n = \ell m^2 \equiv 7 \pmod 8$. Then the M_{24}-representation K_n determined by the coefficients of $q^{n/8}$ of the McKay-Thompson series contains the following conjugate pairs of irreducible representations with odd (and therefore positive) multiplicity:*

- *For $\ell = 7$, the total number of pairs (χ_3, χ_4) and (χ_{12}, χ_{13});*
- *for $\ell = 7$ and m divisible by 3, the pair (χ_{15}, χ_{16});*
- *for $\ell = 15$, the pair (χ_5, χ_6);*
- *for $\ell = 23$, the pair (χ_{10}, χ_{11}).*

Proof. Let

$$K_n = \bigoplus_{i=1}^{26} m_{\chi_i}\, \chi_i \tag{1}$$

be the decomposition of the M_{24}-representation K_n into its irreducible constituents, i.e. m_{χ_i} is the multiplicity in which χ_i occurs in K_n. We note that if K_n is non-zero then n is odd, i.e. we only need to consider the cases $n = \ell m^2$ with m odd.

First, we consider the cases $\ell = 15$ and $\ell = 23$ and take on both sides of (1) the trace of an element of g of type $15A$ or $23A$, respectively. If n is of the form ℓm^2 with m odd, the left-hand side is odd by Theorem 1.2. For the right-hand side an inspection of the character table of M_{24} shows that if $\lambda \neq \bar{\lambda}$ one has that $\mathrm{Tr}(g|\lambda) = \mathrm{Tr}(g|\bar{\lambda})$ is integral unless $(\lambda, \bar{\lambda}) = (\chi_5, \chi_6)$ or (χ_{10}, χ_{11}), respectively, in which case $\mathrm{Tr}(g|\lambda + \bar{\lambda}) = -1$. Using Theorem 1.1, it follows that $m_\lambda = m_{\bar{\lambda}}$ has to be odd.

For the cases $\ell = 7$, we take in (1) the trace of an element of type $7A$. Here, only the characters (χ_3, χ_4) or (χ_{12}, χ_{13}) can provide an odd contribution to the right-hand side of (1) and so in total an odd number of those pairs has to appear. For m divisible by 3, we take in addition the trace of an element of type $21A$. Here (χ_3, χ_4), (χ_{12}, χ_{13}) and (χ_{15}, χ_{16}) provide an odd contribution. Since the total number of pairs of type (χ_3, χ_4) and (χ_{12}, χ_{13})

is odd and by Theorem 1.2 the left-hand side of (1) is even, it follows that the number of pairs (χ_{15}, χ_{16}) is odd, too. $\square$

We remark that the methods in proving Theorem 1.2 and Theorem 1.3 can also be applied to the other cases of umbral moonshine.

The paper is organized as follows. The case $1A$ is clear since $\Sigma_{1A}(\tau) = \Sigma(\tau)$ and equation (3). It is trivial to see that for the cases $2B$, $3B$, $4A$, $4C$, $6B$, $10A$, $12A$, $12B$, the coefficients of $\Sigma_g(\tau)$ are even because $\Sigma_g(\tau)$ is -2 times an η-product (see Appendix). Therefore, we omit the study of these cases. In Section 2, we study the cases $7A$, $14A$, $15A$, $21A$, $23A$. In Section 3, we study the remaining cases.

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2 The odd cases

Let $\ell X \in \{7A, 14A, 15A, 23A\}$, then we show that the coefficient of $q^{n/8}$ in $\Sigma_{\ell X}(\tau)$ is odd if and only if $n = \ell m^2$, where m is odd. For $\ell X = 21A$ this coefficient is odd if and only if $n = \ell m^2$, where m is odd and m is not divisible by 3.

We let

$$\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)$$

be the Dedekind η-function and consider for $N \geq 2$ the Eisenstein series

$$\begin{aligned} \phi_2^{(N)}(\tau) &= \frac{24}{N-1}\, q \frac{d}{dq} \log\left(\frac{\eta(N\tau)}{\eta(\tau)}\right) \\ &= 1 + \frac{24}{N-1} \sum_{k=1}^{\infty} \sigma_1(k)(q^k - Nq^{Nk}) \end{aligned} \tag{2}$$

of weight 2 for $\Gamma_0(N)$. Moreover, for the case $23A$, we use functions $f_{23,1}(\tau)$ and $f_{23,2}(\tau)$ which are modular forms for $\Gamma_0(23)$, explicitly given in [8, Appendix A.1].

We will discuss the case of $7A$ in detail. Since the other cases can be handled in complete analogy, we collect the necessary data in Table 1.

Proof. Let

$$E_2(\tau) = 1 - 24\sum_{m=1}^{\infty} \sigma_1(m) q^m.$$

By [6], we have

$$\Sigma(\tau) = -2\left(E_2(\tau) + 24\sum_{n=1}^{\infty} \frac{(-1)^n n q^{\frac{1}{2}n(n+1)}}{1-q^n}\right) \Big/ \eta(\tau)^3 . \tag{3}$$

Thus

$$\begin{aligned}\Sigma_{7A}(\tau) &= \frac{1}{8}\left(\Sigma(\tau)\eta(\tau)^3 - 14\,\phi_2^{(7)}(\tau)\right)\Big/\eta(\tau)^3 \\ &= -\frac{1}{4}\left(E_2(\tau) + 24\sum_{n=1}^{\infty}\frac{(-1)^n n q^{\frac{1}{2}n(n+1)}}{1-q^n} + 7\phi_2^{(7)}(\tau)\right)\Big/\eta(\tau)^3 .\end{aligned}$$

Define the function $f_m(\tau)$ as follows:

$$f_m(\tau) = \frac{1}{4}\Big(\vartheta_3(\frac{m\tau}{8}) - \vartheta_4(\frac{m\tau}{8})\Big)$$

where $\vartheta_3(\tau) = 1 + \sum_{m=1}^{\infty} 2q^{m^2}$ and $\vartheta_4(\tau) = 1 + \sum_{m=1}^{\infty} 2(-q)^{m^2}$. Then

$$\begin{aligned} f_7(\tau) &= \frac{1}{4}\Big(\vartheta_3(\frac{7\tau}{8}) - \vartheta_4(\frac{7\tau}{8})\Big) = q^{7/8} + q^{63/8} + q^{175/8} + q^{343/8} + \cdots , \\ \eta(\tau)^3 f_7(\tau) &= q - 3q^2 + 5q^4 - 7q^7 + q^8 - 3q^9 + 14q^{11} + \cdots \end{aligned}$$

We call $f_7(\tau)$ the "characteristic function."

We have to show that all coefficients of $\Sigma_{7A}(\tau) + f_7(\tau)$ are even. This follows if all coefficients of $(\Sigma_{7A}(\tau) + f_7(\tau))\eta(\tau)^3$ are even.

First, we observe that the constant term is even. Since all coefficients, except for the constant term, of the function

$$-\frac{1}{4}\Big(E_2(\tau) + 24\sum_{n=1}^{\infty}\frac{(-1)^n n q^{\frac{1}{2}n(n+1)}}{1-q^n}\Big)$$

are even, it is enough to show that the coefficients of the following function, again except for the constant term,

$$-\frac{7}{4}\phi_2^{(7)}(\tau)+\eta(\tau)^3 f_7(\tau)=-\frac{7}{4}-6q-24q^2-28q^3-44q^4-42q^5+\cdots.$$

are even. Define the "sieve function"

$$\frac{7}{4}\vartheta_3(\tau)^4=\frac{7}{4}+14q+42q^2+56q^3+42q^4+84q^5+\cdots.$$

The coefficients of $\frac{7}{4}\vartheta_3(\tau)^4$ without the constant term are even, hence it is enough to show that the coefficients of the function

$$-\frac{7}{4}\phi_2^{(7)}(\tau)+\eta(\tau)^3 f_7(\tau)+\frac{7}{4}\vartheta_3(\tau)^4=8q+18q^2+28q^3-2q^4+42q^5+\cdots$$

are even. We are going to prove this using Sturm's Theorem. The theta functions $\vartheta_3(\tau)$ and $\vartheta_4(\tau)$ can be expressed as a quotient of η-functions, namely

$$\vartheta_3(\tau)=\frac{\eta(2\tau)^5}{\eta(\tau)^2\eta(4\tau)^2}\qquad\text{and}\qquad\vartheta_4(\tau)=\frac{\eta(\tau)^2}{\eta(2\tau)}.$$

It follows using [17, Theorem 1.64] that $\eta(8\tau)^3 f_7(8\tau)$ is a modular form of weight 2 for $\Gamma_0(448)$ and hence

$$\begin{aligned}\sum_{m=1}^{\infty}a_{7A}(m)q^m&:=-\frac{7}{4}\phi_2^{(7)}(8\tau)+\eta(8\tau)^3 f_7(8\tau)+\frac{7}{4}\vartheta_3(8\tau)^4\\&=8q^8+18q^{16}+28q^{24}+\cdots\end{aligned}$$

is also a modular form of weight 2 for $\Gamma_0(448)$. Using Sturm's theorem [17, Theorem 2.58] and the fact that $[SL_2(\mathbb{Z}):\Gamma_0(448)]=768$, the computer verification that $a_{7A}(m)\equiv 0 \pmod 2$ for $m\leq 129$ shows $a_{7A}(m)\equiv 0 \pmod 2$ for all m.

This completes the proof of the case $7A$. □

The proof for the other cases is analogous. The relevant information can be read off from Table 1.

Table 1: Data for the proofs in Section 2

ℓX	characteristic functions	sieve functions	Γ	$[SL_2(\mathbb{Z}):\Gamma]$	Sturm bounds
$7A$	$f_7(\tau)$	$\frac{7}{4}\vartheta_3(\tau)^4$	$\Gamma_0(448)$	768	129
$14A$	$f_7(\tau)$	$\frac{23}{12}\phi_2^{(2)}(\tau)$	$\Gamma_0(448)$	768	129
$15A$	$f_{15}(\tau)$	$2\vartheta_3(\tau)^4$	$\Gamma_0(960)$	2304	385
$21A$	$f_7(\tau)$-$f_{63}(\tau)$	$\frac{23}{12}\phi_2^{(2)}(\tau)$	$\Gamma_0(4032)$	9216	1537
$23A$	$f_{23}(\tau)$	$\frac{23}{12}\phi_2^{(2)}(\tau)$	$\Gamma_0(1472)$	2304	385

3 The remaining even cases

In this section we prove that all coefficients for the remaining cases $\ell X \in \{2A, 3A, 4B, 5A, 6A, 8A, 11A\}$ are divisible by two.

3.1 The cases $2A$, $3A$, $4B$, $5A$, $6A$, $8A$

We give the detailed proof for the case of $2A$. The five other cases are proved analogously.

Proof. The McKay-Thompson series for $2A$ is

$$\begin{aligned}\Sigma_{2A}(\tau) &= \frac{1}{3}\left(\Sigma(\tau)\eta(\tau)^3 - 4\phi_2^{(2)}(\tau)\right)\Big/\eta(\tau)^3 \\ &= -2q^{-1/8} - 6q^{7/8} + 14q^{15/8} - 28q^{23/8} + 42q^{31/8} + \cdots .\end{aligned}$$

The coefficients of $\Sigma(\tau)\eta(\tau)^3$, except for the constant term, are divisible by 24 (cf. (3)). Moreover, also the coefficients of $\phi_2^{(2)}(\tau)$, again except for the constant term, are divisible by 24. The constant term of the function $\Sigma(\tau)\eta(\tau)^3 - 4\phi_2^{(2)}(\tau)$ is 6. Therefore, the coefficients of $\Sigma_{2A}(\tau)$ are divisible by two. $\square$

The proof for the other cases is analogous. The relevant information can be read off from equation (2) together with the appendix.

3.2 The case $11A$

Finally, it remains to show that the Fourier coefficients of the McKay-Thompson series for the case $11A$ are divisible by two.

Proof. The McKay-Thompson series for $11A$ is

$$\Sigma_{11A}(\tau) = \frac{1}{12}\left(\Sigma(\tau)\eta(\tau)^3 - 22\,\phi_2^{(11)}(\tau) + \frac{264}{5}(\eta(\tau)\eta(11\tau))^2\right)\Big/\eta(\tau)^3\,.$$

The coefficients of $\Sigma(\tau)\eta(\tau)^3$, except for the constant term are divisible by 24. We need to prove that the coefficients of the function:

$$22\,\phi_2^{(11)}(\tau) - \frac{264}{5}(\eta(\tau)\eta(11\tau))^2 = 22 + 264q^2 + 264q^3 + 264q^4 + \cdots$$

are also (except for the constant term) divisible by 24. The coefficients of $22\phi_2^{(2)}(\tau)$ (except for the constant term) are divisible by 24, hence it is enough to show that the coefficients of the function

$$\sum_{m=1}^{\infty} a_{11A}(m)q^m := 22\phi_2^{(11)}(\tau) - \frac{264}{5}(\eta(\tau)\eta(11\tau))^2 - 22\,\phi_2^{(2)}(\tau)$$
$$= -528q - 264q^2 - 1848q^3 - 264q^4 - 2904q^5 - 1584q^6 - 3696q^7 + \cdots$$

are divisible by 24. This is true, since this function is a modular form for $\Gamma_0(22)$ [13]. Using Sturm's theorem [17, Theorem 2.58] and the fact that $[SL_2(\mathbb{Z}) : \Gamma_0(22)] = 36$, the verification that $a_{11A}(m) \equiv 0 \pmod{24}$ for $m \leq 7$ shows $a_{11A}(m) \equiv 0 \pmod{24}$ for all m. Therefore, the coefficients of $\Sigma_{11A}(\tau)$ are divisible by two. $\square$

Sections 2 and 3 together with the observations at the end of the introduction prove Theorem 1.2.

A McKay-Thompson series

We list all McKay-Thompson series using the ATLAS [4] names for the conjugacy classes of M_{24}. We refer to Section 2 for the used functions.

$$\mathbf{1A}: \quad \Sigma_{1A}(\tau) = \Sigma(\tau)$$

$$\mathbf{2A}: \quad \Sigma_{2A}(\tau) = \tfrac{1}{3}\left(\Sigma(\tau)\eta(\tau)^3 - 4\,\phi_2^{(2)}(\tau)\right)\Big/\eta(\tau)^3$$

$$\mathbf{2B}: \quad \Sigma_{2B}(\tau) = -2\;\eta(\tau)^5/\eta(2\tau)^4$$

$$\mathbf{3A}: \quad \Sigma_{3A}(\tau) = \tfrac{1}{4}\left(\Sigma(\tau)\eta(\tau)^3 - 6\,\phi_2^{(3)}(\tau)\right)\Big/\eta(\tau)^3$$

$$\mathbf{3B}: \quad \Sigma_{3B}(\tau) = -2\;\eta(\tau)^3/\eta(3\tau)^2$$

$$\mathbf{4A}: \quad \Sigma_{4A}(\tau) = -2\;\eta(2\tau)^8\big/\big(\eta(\tau)^3\eta(4\tau)^4\big)$$

$$\mathbf{4B}: \quad \Sigma_{4B}(\tau) = \tfrac{1}{6}\left(\Sigma(\tau)\eta(\tau)^3 + 2\phi_2^{(2)}(\tau) - 12\,\phi_2^{(4)}(\tau)\right)\Big/\eta(\tau)^3$$

$$\mathbf{4C}: \quad \Sigma_{4C}(\tau) = -2\;\eta(\tau)\eta(2\tau)^2/\eta(4\tau)^2$$

$$\mathbf{5A}: \quad \Sigma_{5A}(\tau) = \tfrac{1}{6}\left(\Sigma(\tau)\eta(\tau)^3 - 10\,\phi_2^{(5)}(\tau)\right)\Big/\eta(\tau)^3$$

$$\mathbf{6A}: \quad \Sigma_{6A}(\tau) = \tfrac{1}{12}\left(\Sigma(\tau)\eta(\tau)^3 + 2\,\phi_2^{(2)}(\tau) + 6\,\phi_2^{(3)}(\tau) - 30\,\phi_2^{(6)}(\tau)\right)\Big/\eta(\tau)^3$$

$$\mathbf{6B}: \quad \Sigma_{6B}(\tau) = -2\;\eta(2\tau)^2\eta(3\tau)^2\big/\big(\eta(\tau)\eta(6\tau)^2\big)$$

$$\mathbf{7A}: \quad \Sigma_{7A}(\tau) = \tfrac{1}{8}\left(\Sigma(\tau)\eta(\tau)^3 - 14\,\phi_2^{(7)}(\tau)\right)\Big/\eta(\tau)^3$$

$$\mathbf{8A}: \quad \Sigma_{8A}(\tau) = \tfrac{1}{12}\left(\Sigma(\tau)\eta(\tau)^3 + 6\,\phi_2^{(4)}(\tau) - 28\,\phi_2^{(8)}(\tau)\right)\Big/\eta(\tau)^3$$

$$\mathbf{10A}: \quad \Sigma_{10A}(\tau) = -2\;\eta(2\tau)\eta(5\tau)/\eta(10\tau)$$

$$\mathbf{11A}: \quad \Sigma_{11A}(\tau) = \tfrac{1}{12}\left(\Sigma(\tau)\eta(\tau)^3 - 22\,\phi_2^{(11)}(\tau) + \tfrac{264}{5}\,(\eta(\tau)\eta(11\tau))^2\right)\Big/\eta(\tau)^3$$

$$\mathbf{12A}: \quad \Sigma_{12A}(\tau) = -2\;\eta(4\tau)^2\eta(6\tau)^3\big/\big(\eta(2\tau)\eta(3\tau)\eta(12\tau)^2\big)$$

$$\mathbf{12B}: \quad \Sigma_{12B}(\tau) = -2\;\eta(\tau)\eta(4\tau)\eta(6\tau)\big/\big(\eta(2\tau)\eta(12\tau)\big)$$

$$\begin{aligned}\mathbf{14A}: \quad \Sigma_{14A}(\tau) = \tfrac{1}{24}\Big(&\Sigma(\tau)\eta(\tau)^3 + \tfrac{2}{3}\,\phi_2^{(2)}(\tau) + 14\phi_2^{(7)}(\tau) - \tfrac{182}{3}\,\phi_2^{(14)}(\tau) + \\ &+112\,\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)\Big)\Big/\eta(\tau)^3\end{aligned}$$

$$\begin{aligned}\mathbf{15A}: \quad \Sigma_{15A}(\tau) = \tfrac{1}{24}\Big(&\Sigma(\tau)\eta(\tau)^3 + \tfrac{3}{2}\,\phi_2^{(3)}(\tau) + 5\,\phi_2^{(5)}(\tau) - \tfrac{105}{2}\,\phi_2^{(15)}(\tau) + \\ &+90\,\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)\Big)\Big/\eta(\tau)^3\end{aligned}$$

$$\mathbf{21A}: \quad \Sigma_{21A}(\tau) = -\tfrac{1}{3}\left(7\,\eta(7\tau)^3\big/\big(\eta(3\tau)\eta(21\tau)\big) - \eta(\tau)^3/\eta(3\tau)^2\right)$$

$$\mathbf{23A}: \quad \Sigma_{23A}(\tau) = \tfrac{1}{24}\left(\Sigma(\tau)\eta(\tau)^3 - 46\,\phi_2^{(23)}(\tau) + \tfrac{276}{11}\,f_{23,1}(\tau) + \tfrac{1932}{11} f_{23,2}(\tau)\right)\Big/\eta(\tau)^3$$

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